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Slip Flow and Hypersonic Boundary Layers

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The problem of the hypersonic similar laminar boundary layeı is modified to include first ordeı slip boundary conditions It is observed that the inclusion of these new boundary conditions result in no change in the energy transfer to the wall from the no-slip values, implying that slip effects alone are not adequate to explain the appearance of the heat tıansfer "plateaus" observed near the leading edge of a sharp flat plate in hypersonic flow In addition, it is shown how the inclusion of slip boundary condition as the only higher order effect results in complete disagreement with observed pressure measurements for the cold-wall case

In a recent note, Talbot[1] suggests a criterion for slip flow near the sharp leading edge of a flat plate in hypersonic flow It is significant that the parameter suggested by Talbot as a suitable correlating parameter for heat transfer and induced pressure experiments can be interpreted both as the ratio of the mean free path behind the shock wave to the boundary layer thickness, as well as the ratio of the leading edge shock thickness to the distance from the leading edge Although Talbot's correlating parameter is not really the appropriate one which arises from the inclusion of slip boundary conditions to the traditional strong hypersonic viscous interaction problem, it can, of course, be related to that parameter The author has included first-order slip boundary conditions in the calculation of heat transfer for that case and has arrived at the not unexpected conclusion that there is no effect of hypersonic laminar heat transfer of the inclusion of slip velocity and temperature jump boundary conditions This result, which was obtained earlier by Maslen[2] for a zero-pressure gradient no-slip flow is here extended to the class of hypersonic similar solutions characterized by u = const, $p \sim x^n$ These solutions, which were tabulated by Dewey,[3] include the conventional no-slip strong interaction problem where $p \sim x^{-1/2}$ Our negative result is restricted only to the heat transfer to the wall Induced pressure and skin friction, on the other hand, in general will exhibit a first-order slip effect Thus the possibility is raised that additional higher order effects besides slip will be necessary to account for the observed pressure and heat transfer "plateaus "

At the present time, it is not clear just how a systematic approach to the solution of the Navier-Stokes equation should proceed for this geometry, and there is even the possibility that coupling between the viscous boundary layer and the shock structure may not be very well represented by the Navier-Stokes equation It is felt that it is inappropriate to attribute to slip flow and rarefaction effects which may well be within the scope of the Navier-Stokes equations

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It is quite apparent from our results that slip flow alone is incapable of reducing the energy transfer to the wall from the usual strong interaction result Perhaps the success of Talbot's parameter in correlating both induced pressure and heat transfer measurements hinges on its dual significance as an estimator of the effects of shock-structure viscous flow field coupling, as well as an estimator of slip effects

Analysis

Consider a perfect gas, constant Prandtl number, and arbitrary viscosity-temperature dependence The flow external to the boundary layer is presumed to be hypersonic, corresponding to the limit $U\delta^2/2H\delta \to 1$ The notation of Hayes and Probstein[4] is used throughout

The form of the first-order slip boundary conditions in the η,ξ variables suggests that the appropriate expansion parameter for first-order slip effects is $\epsilon \equiv \lambda_w \rho_w U\delta/(2\xi_0)^{1/2}$ For the hypersonic no-slip similar flows corresponding to $U\delta$ = const, $p_0 \sim x^n$, this parameter ϵ varies as $x^{-(n+1/2)}$ Consider a first-order expansion in ϵ of f, g, p ξ, and $\rho\mu/\rho_w\mu_w$:

$$\left.\begin{aligned} f &= f_0(\eta) + \epsilon f_1(\eta) \\ g &= g_0(\eta) + \epsilon g_1(\eta) \\ p &= p_0[1 + \epsilon p_1] \\ \xi &= \xi_0[1 + 2\epsilon p_1] \\ \rho\mu/\rho_w\mu_w &\equiv (\mu/T)\ (T_w/\mu_w) \equiv N \\ N &= N_0 + \epsilon \frac{T_w}{g_w}\frac{dN}{dT}\Big)_{T=T_0} (g_1 - 2f_1'f_0') \end{aligned}\right\} \qquad (1)$$

If the effect of the self-induced pressure gradient is to be calculated, then p_1 will be determined by a suitable relationship between the displacement thickness and the pressure Otherwise it may be set identically equal to zero In the analysis of the heat transfer, which will be given below, it is quite apparent that its magnitude does not enter into consideration at all It does, however, enter into the momentum equation as a coefficient of an inhomogeneous term and consequently will be involved in the skin friction and in the solution for f_1 and g_1

The first-order boundary layer equations with appropriate boundary conditions become

$$\frac{\partial}{\partial\eta}\left\{N_0 f_1'' + \frac{T_w}{g_w}\left(\frac{dN}{dT}\right)_{T=T_0}(g_1 - 2f_0'f_1')\,f_0''\right\} + f_1 f_0'' + f_0 f_1'' + \beta_0[g_1 - 2f_0'f_1'] + \beta_1\beta_0(g_0 - f_0')^2 = \left[\beta_0\left(\frac{h\delta_0}{H\delta}\right) - 1\right][f_1'f_0' - f_1 f_0''] \qquad (2a)$$

$$\frac{1}{Pr}\frac{\partial}{\partial\eta}\left\{N_0 g_1' + \frac{T_w}{g_w}\frac{dN_0}{dT}(g_1 - 2f_0'f_1')g_0'\right\} + f_0 g_1' + f_1 g_0' + 2\frac{\partial}{\partial\eta}\left\{N_0\left(1 \frac{1}{Pr}\right)(f_1'f_0'' + f_0'f_1'')\right\} + 2\frac{\partial}{\partial\eta}\left\{\frac{T_w}{g_w}\left(\frac{dN_0}{dT}\right)_{T=T_0}(g_1 - 2f_0'f_1')\left(1 - \frac{1}{Pr}\right)(f_0'f_0'')\right\} = \left[\beta_0\left(\frac{h\delta_0}{H\delta}\right) - 1\right][g_1 f_0' - f_1 g_0'] \qquad (2b)$$

The boundary conditions are $f_1'(0) = a_v f_0''(0)$, $g_1(0) = a_T g_0'$, where a is a velocity slip coefficient, a_T is a temperature jump coefficient, $f_1(0) = 0$, $f_1'(\infty) \to 1$, $g_1(\infty) \to 0$ Note that

$$\frac{2d\ln U_\infty}{d\ln\xi}\frac{H\delta}{h\delta} \equiv \beta_0(1 + \epsilon\beta_1) = \frac{\gamma - 1}{\gamma}\quad\frac{n}{n-1}\left(1 + \epsilon\frac{3}{2}p_1\right) \qquad (3)$$

In the hypersonic limit, terms of $0[\gamma - 1/\gamma]$ $[h\delta/H\delta]$ may be neglected with respect to terms of order 1 The energy equation can now be written

$$\frac{1}{Pr}\frac{\partial}{\partial\eta}\left\{N_0 g_1' + \frac{T_w}{g_w}\frac{dN}{dT}(g_1 - 2f_0'f_1')g_0'\right\} + (f_0 g_1)' + \frac{2\partial}{\partial\eta}\left\{N_0\left(1 - \frac{1}{Pr}\right)(f_1'f_0'' + f_0'f_1'')\right\} + 2\frac{\partial}{\partial\eta}\left\{\frac{T_w}{g_w}\frac{dN}{dT}(g_1 - 2f_0'f_1')(f_0'f_0'')\right\} = 0 \quad (4)$$

Equation (4) can be integrated and the constant of integration set equal to zero in order to insure the exponential decay of the profiles, as well as the boundedness of the various boundary layer thicknesses The resulting equation can be evaluated at $\eta = 0$:

$$\frac{1}{Pr}\left\{g_1'(0) + \left(\frac{T_w}{g_w}\frac{dN}{dT}\right)_{T=T_w}[g_1(0)g_0'(0)]\right\} + 2\left(1 - \frac{1}{Pr}\right)[f_1'(0)f_0''(0)'] = 0 \quad (5)$$

The energy transfer to a wall in a slip flow is $[k(\partial T/\partial y)_0] + (\tau u)_0 \equiv q$ In terms of our η, ξ variables, this corresponds to

$$q = \frac{H_\delta\rho(0)\mu(0)\mu_\delta}{(2\xi)^{1/2}Pr}[g(0) - 2f'(0)f''(0)] + \frac{H_\delta\rho(0)\mu(0)2f'(0)f''(0)}{(2\xi)^{1/2}} \quad (6)$$

In terms of our expansion parameter ϵ,

$$q = \frac{H_\delta p_0(1 + \epsilon p_1)(1 - \epsilon p_1)}{2\xi_0^{1/2}Pr}\frac{\mu_w}{RT_w}\left[1 + \left(\frac{T_w}{g_w}\frac{dN}{dT}\right)_{T=T_w}(g_1)\epsilon\right] \times [g_0'(0) + \epsilon g_1'(0) - 2\epsilon f_1'(0)f_0''(0) + 2Pr\epsilon f_1'(0)f_0''(0)] \quad (7a)$$

or

$$q = q_0\left[1 + \epsilon\left(\frac{T_w}{g_w}\frac{dN}{dT}\right)_{T=T_w} g_1(0) + \epsilon\frac{g_1'(0)}{g_0'(0)} + 2(Pr - 1)\epsilon \times \frac{f_1'(0)f_0''(0)}{g_0'(0)}\right] \quad (7b)$$

From Eq (5), the coefficient of ϵ is equal to 0 Thus $q_{\text{lip}} = q_{\text{no slip}}$

As noted earlier, this result has been obtained before by Maslen[2] and has been the center of some controversy among Rott,[5] Shen,[6] and Maslen[7] It would appear that this especially simple result, independent of the magnitudes of Pr, a_v, or a_T, hinges on the correct application of the transformation $(\partial\eta/\partial_y)_{y=0} = \rho(0)u_\delta/(2\xi)^{1/2}$ rather than $\rho(w)u_\delta/2^{1/2}\xi$ For $(\gamma - 1)/\gamma \to 0$, the solutions of Eqs (2a) and (2b) are

$$f_1' = f_0''(\eta)\left\{a_V - [a_T - a_V]g_0'(0)\int_0^\eta\left[\frac{N(\eta)f_0''(\eta)}{N(0)f_0''(0)}\right]^{Pr} \times \left[\frac{dN_0}{N_0 dT_0}\frac{T_w}{g_w}\right]d\eta\right\} \quad (8a)$$

$$g_1 = g_0'(\eta)\left\{a_V - [a_T - a_V]g_0'(0)\int_0^\eta\left[\frac{N_0(\eta)f''(\eta)}{N_0(0)f''(0)}\right]^{Pr} \times \left[\frac{dN_0}{N_0 dT_0}\right]\frac{T_w}{g_w}d\eta\right\} + [a_T - a_V]g_0'(0)\left[\frac{N_0(\eta)f_0''(\eta)}{N(0)f_0''(0)}\right]^{Pr} \quad (8b)$$

An approximate estimate of the effect of slip on induced pressure can be obtained quite simply for this limiting hypersonic case, $N = 1$ and $Pr = 1$ The displacement thicknes- can be written

$$\delta^* = \delta_0^*[1 + \epsilon\delta_1/\delta_0] \quad (9a)$$

where

$$\delta_1 = \int_0^\infty (g_1 - 2f_0'f_1')d\eta \quad (9b)$$

$$\delta_0 = \int_0^\infty (g_0 - f_0'^2)d\eta \quad (9c)$$

Thus

$$\delta_1 \approx -a_V g_w + [a_T - a\ ][1 - g_w] \quad (10)$$

From the tangent wedge formula, $p_1 = \frac{4}{3}(\delta_1/\delta_0)$, or using the Crocco integral and the Blasius' results,

$$p_1 \approx \frac{2^{1/2}(-a_v g_w + [a_T - a_v][1 - g_w])}{1\ 73 g_w + 0\ 664}\frac{4}{3} \quad (11)$$

This result implies that slip effects on a cold wall will tend to increase the induced pressure from the no-slip values, whereas slip effects on a near adiabatic wall, where $g_w \sim 1$, will lower the induced pressure The latter effect has been observed, while the former disagrees with currently available data Our analysis, although incomplete and approximate, does suggest that slip alone is not sufficient to cause strong interaction theory to agree with experiment as the leading edge is approached, at least for a cold wall

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Reply by Author to J Aroesty

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THE writer does not think that there are any differences of opinion between Aroesty and himself, except perhaps in what we imply in our use of words like "slip flow" and "rarefaction effects," and certainly he agrees with him that the leading edge region cannot be treated adequately merely by applying slip boundary conditions to the hypersonic boundary layer equations The main purpose of the note was to suggest that the region of the leading edge flow, where the predictions of hypersonic boundary layer theory begin to depart from the experiment, might be characterized by a particular Knudsen number based on properties of the hypersonic boundary layer The fluid-dynamic phenomena responsible for these departures most likely include the merging of a thick shock wave with a completely viscous layer, as well as slip at the wall, and the flow probably resembles more the model considered by Oguchi[1] than a slipping hypersonic boundary

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